

# ASYMPTOTIC BEHAVIOR OF THE LEAST COMMON MULTIPLE OF CONSECUTIVE REDUCIBLE QUADRATIC PROGRESSION TERMS

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ABSTRACT. Let  $l$  and  $m$  be two integers with  $l > m \geq 0$ , and let  $f(x)$  be the product of two linear polynomials with integer coefficients. In this paper, we show that  $\log \text{lcm}_{mn < i \leq ln} \{f(i)\} = An + o(n)$ , where  $A$  is a constant depending only on  $l$ ,  $m$  and  $f$ .

## 1. Introduction

The study of the least common multiple of consecutive positive integers was first initiated by Chebyshev for a significant attempt to prove prime number theorem. From Chebyshev's well-known work [2], one can easily get an equivalent of prime number theorem which states that  $\log \text{lcm}(1, \dots, n) \sim n$  as  $n$  tends to infinity. Since then, this topic received attentions of many authors. Hanson [5] and Nair [15] got the upper and lower bound of  $\text{lcm}_{1 \leq i \leq n} \{i\}$ , respectively. Bateman, Kalb and Stenger [1] gave an asymptotic formula of  $\log \text{lcm}_{1 \leq i \leq n} \{b + ai\}$  as  $n$  tends to infinity, where  $a$  and  $b$  are coprime integers. Farhi [3], Hong and Feng [6], Hong and Yang [12], Hong and Kominers [7], Wu, Tan and Hong [20] and Kane and Kominers [13] obtained lower bounds of the least common multiple of the first  $n$  arithmetic progression terms. Farhi and Kane [4] studied the least common multiple of consecutive integers. Hong and Qian [9] obtained some results on the least common multiple of consecutive arithmetic progression terms which was consequently extended in one direction by Qian, Tan and Hong [19]. Hong, Qian and Tan [11] got an asymptotic formula of the least common multiple of a sequence of products of linear polynomials. On the other hand, Farhi [3] obtained a nontrivial lower bound for the least common multiple of the quadratic sequence  $\{i^2 + 1\}_{i=1}^{\infty}$ . Oon [16] improved some of the Hong-Kominers result and Farhi's lower bound. Hong, Luo, Qian and Wang [8] extended Nair's and Oon's lower bound by giving a uniform lower bound. Qian, Tan and Hong [18] showed that for any given positive integer  $k$ , we have  $\log \text{lcm}_{0 \leq i \leq k} \{(n+i)^2 + 1\} \sim 2(k+1) \log n$  as  $n \rightarrow \infty$ . Recently, Hong and Qian [10] got some interesting results on the least common multiple of consecutive quadratic progression terms.

Qian and Hong [17] investigated the asymptotic behavior of the least common multiple of any consecutive arithmetic progression terms. Let  $l$  and  $m$  be integers with  $l > m \geq 0$

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and let  $a \geq 1$  and  $b$  be integers such that  $a + b \geq 1$  and  $\gcd(a, b) = 1$ . It is proved in [17] that

$$\log \text{lcm}_{mn < i \leq ln} \{ai + b\} \sim \frac{an}{\varphi(a)} \sum_{\substack{r=1 \\ \gcd(r, a)=1}}^a B_r$$

as  $n \rightarrow \infty$ , where

$$(1.1) \quad B_r := \begin{cases} \frac{l}{r}, & \text{if } l \geq \frac{(a+r)m}{r}, \\ \sum_{i=0}^{\mathcal{K}-1} \frac{l-m}{r+ai} + \frac{l}{r+ai\mathcal{K}}, & \text{if } l < \frac{(a+r)m}{r} \end{cases}$$

with  $\mathcal{K} := \lfloor \frac{al-(l-m)r}{a(l-m)} \rfloor$  and  $\lfloor x \rfloor$  being the largest integer no more than  $x$ .

In this paper, we mainly concentrate on the asymptotic behavior of the least common multiple of consecutive reducible quadratic progression terms. There are two cases about the reducible quadratic progressions. The first case is  $f(x) = (ax + b)^2$  with  $a \geq 1$  and  $b$  being integers such that  $a + b \geq 1$  and  $\gcd(a, b) = 1$ . This case is easy to answer. Actually, by the main result of [17], we can derive immediately that

$$\log \text{lcm}_{mn < i \leq ln} \{(ai + b)^2\} \sim \frac{2an}{\varphi(a)} \sum_{\substack{r=1 \\ \gcd(r, a)=1}}^a B_r$$

as  $n \rightarrow \infty$ , where  $B_r$  is defined as in (1.1).

Our main goal in the present paper is to treat with the second case that  $f(x) = (a_1x + b_1)(a_2x + b_2)$  with  $a_i, b_i \in \mathbb{N}^*$  and  $\gcd(a_i, b_i) = 1$  for  $1 \leq i \leq 2$  and  $a_1b_2 \neq a_2b_1$ . Let  $\mathbb{N}$  be the set of nonnegative integers and  $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ . For any two positive integers  $a$  and  $b$ , let  $\langle b \rangle_a$  denote the smallest positive integer congruent to  $b$  modulo  $a$  between 1 and  $a$ . For any integer  $t$ , we define  $S_t$  by  $S_t := \{i \in \mathbb{N} : 0 \leq i \leq t\}$ . Clearly,  $S_t$  is empty if  $t$  is negative and so we can define  $\sum_{i \in S_t} g(i) := 0$  for any arithmetic function  $g$  if  $t < 0$ . We define the following three 4-variable arithmetic functions:

$$(1.2) \quad g_r(x, y, z, w) := \left\lfloor \frac{xy\langle l \rangle_x + ym\langle zr \rangle_x - xl\langle wr \rangle_y}{xy(l-m)} \right\rfloor,$$

$$(1.3) \quad h_r(x, y, z, w) := \left\lfloor \frac{xm\langle wr \rangle_y - yl\langle zr \rangle_x}{xy(l-m)} \right\rfloor$$

and

$$(1.4) \quad \lambda_r(x, y, z, w) := \sum_{i \in S_{g_r(x, y, z, w)}} \frac{xl}{\langle zr \rangle_x + xi} - \sum_{i \in S_{g_r(x, y, z, w)-1}} \frac{ym}{\langle wr \rangle_y + yi} \\ + \sum_{i \in S_{h_r(x, y, z, w)}} \left( \frac{yl}{\langle wr \rangle_y + yi} - \frac{xm}{\langle zr \rangle_x + xi} \right).$$

We can now state the main result of this paper.

**Theorem 1.1.** *Let  $l$  and  $m$  be fixed integers with  $l > m \geq 0$ . Let  $f(x) = (a_1x + b_1)(a_2x + b_2)$ , where  $a_i, b_i \in \mathbb{N}^*$  and  $\gcd(a_i, b_i) = 1$  for  $1 \leq i \leq 2$  and  $a_1b_2 \neq a_2b_1$ . Then*

$$\log \text{lcm}_{mn < i \leq ln} \{f(i)\} = \frac{n}{\varphi(q)} \sum_{\substack{r=1 \\ \gcd(r, q)=1}}^q A_r + o(n),$$

where  $q = \text{lcm}(a_1, a_2)$  and

$$(1.5) \quad A_r := \begin{cases} \lambda_r(a_1, a_2, b_1, b_2) & \text{if } a_1 \langle b_2 r \rangle_{a_2} \geq a_2 \langle b_1 r \rangle_{a_1}; \\ \lambda_r(a_2, a_1, b_2, b_1) & \text{if } a_1 \langle b_2 r \rangle_{a_2} < a_2 \langle b_1 r \rangle_{a_1}. \end{cases}$$

Note that Theorem 1.1 is still true if at least one of  $b_1$  and  $b_2$  is a negative integer.

The paper is organized as follows. In Section 2, we prove two lemmas which are needed for the proof of Theorem 1.1. The final section will devote to the proof of Theorem 1.1.

## 2. Two lemmas

In this section, we show two lemmas which are needed in the proof of Theorem 1.1. Throughout, we let

$$(2.1) \quad H_1 := \left\lfloor \frac{a_1 l - (l - m) \langle b_1 r \rangle_{a_1}}{a_1 (l - m)} \right\rfloor$$

and

$$(2.2) \quad H_2 := \left\lfloor \frac{a_2 l - (l - m) \langle b_2 r \rangle_{a_2}}{a_2 (l - m)} \right\rfloor.$$

As usual, for any prime number  $p$ , we let  $v_p$  be the normalized  $p$ -adic valuation on the set of positive integers. Namely, one has  $v_p(a) = s$  if  $p^s \parallel a$ . We begin with the following result.

**Lemma 2.1.** *Let  $l, m, q$  and  $f(x)$  be defined as in Theorem 1.1. Then*

$$\log \text{lcm}_{mn < i \leq ln} \{f(i)\} = \sum_{\substack{r'=1 \\ \gcd(r', q)=1}}^q \sum_{p \in \mathcal{P}_{r'}} \log p + O(\sqrt{n}),$$

where

$$(2.3) \quad \mathcal{P}_{r'} := \left\{ \text{prime } p : p \equiv r' \pmod{q} \text{ and } p \in \left(0, (l - m)n\right] \bigcup \left( \bigcup_{j=1}^2 \bigcup_{i=0}^{H_j} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \right\}$$

with  $r$  being the unique integer satisfying  $rr' \equiv 1 \pmod{q}$  and  $1 \leq r \leq q$ .

*Proof.* For simplicity, we define  $L_{m,l}^{(f)}(n) := \text{lcm}_{mn < i \leq ln} \{f(i)\}$ , and let  $P_{m,l}^{(f)}(n)$  be the set of all the prime factors of  $L_{m,l}^{(f)}(n)$  not dividing  $\text{lcm}(a_1 b_2 - a_2 b_1, q)$ .

We claim that if  $p \in P_{m,l}^{(f)}(n)$  and  $p | f(i)$  for some integer  $mn < i \leq ln$ , then  $p$  divides exactly one of  $a_1 i + b_1$  and  $a_2 i + b_2$ . Otherwise, we have  $p | (a_1 i + b_1)$  and  $p | (a_2 i + b_2)$ . It implies that  $p \mid (a_1(a_2 i + b_2) - a_2(a_1 i + b_1)) = a_1 b_2 - a_2 b_1$ , which is impossible since  $p \nmid \text{lcm}(a_1 b_2 - a_2 b_1, q)$ . The claim is proved. But the number of prime factors of

$\text{lcm}(a_1b_2 - a_2b_1, q)$  is finite. So we have

$$\begin{aligned}
 (2.4) \quad \log L_{m,l}^{(f)}(n) &= \log \left( \prod_{p \in P_{m,l}^{(f)}(n)} p^{v_p(L_{m,l}^{(f)}(n))} \prod_{p \notin P_{m,l}^{(f)}(n)} p^{v_p(L_{m,l}^{(f)}(n))} \right) \\
 &= \sum_{p \in P_{m,l}^{(f)}(n)} v_p(L_{m,l}^{(f)}(n)) \log p + O(\log(f(ln))) \\
 &= \sum_{p \in P_{m,l}^{(f)}(n)} \log p + \sum_{\substack{p \in P_{m,l}^{(f)}(n) \\ v_p(L_{m,l}^{(f)}(n)) \geq 2}} (v_p(L_{m,l}^{(f)}(n)) - 1) \log p + O(\log n).
 \end{aligned}$$

If  $p \in P_{m,l}^{(f)}(n)$  and  $v_p(L_{m,l}^{(f)}(n)) \geq 2$ , then  $p^2 | f(i)$  for some integer  $i$  with  $mn < i \leq ln$ . Hence by the claim we obtain that  $p^2 | (a_1i + b_1)$  or  $p^2 | (a_2i + b_2)$ , which implies that

$$p \leq M_n := \max\{\sqrt{a_1ln + b_1}, \sqrt{a_2ln + b_2}\} \ll \sqrt{n}.$$

On the other hand, since  $p^{v_p(L_{m,l}^{(f)}(n))} \leq f(ln)$ , it follows that

$$v_p(L_{m,l}^{(f)}(n)) \leq \frac{\log f(ln)}{\log p} \ll \frac{\log n}{\log p}.$$

Hence we get by the prime number theorem that

$$\sum_{\substack{p \in P_{m,l}^{(f)}(n) \\ v_p(L_{m,l}^{(f)}(n)) \geq 2}} (v_p(L_{m,l}^{(f)}(n)) - 1) \log p \ll \sum_{p \leq M_n} \frac{\log n}{\log p} \log p \ll \sum_{p \leq M_n} \log n \ll \frac{\sqrt{n}}{\log \sqrt{n}} \log n \ll \sqrt{n}.$$

It then follows from (2.4) that

$$(2.5) \quad \log L_{m,l}^{(f)}(n) = \sum_{p \in P_{m,l}^{(f)}(n)} \log p + O(\sqrt{n}) + O(\log n) = \sum_{p \in P_{m,l}^{(f)}(n)} \log p + O(\sqrt{n}).$$

First, we give a characterization on the primes in the set  $P_{m,l}^{(f)}(n)$ . By  $T(q)$  we denote the set of all positive integers no more than  $q$  that are relatively prime to  $q$ . Then by the definition of  $P_{m,l}^{(f)}(n)$ , we know that each prime in  $P_{m,l}^{(f)}(n)$  is relatively prime to  $q$ . So each prime  $p \in P_{m,l}^{(f)}(n)$  is congruent to  $r'$  modulo  $q$  for some  $r' \in T(q)$ . For convenience, we let

$$(2.6) \quad \mathcal{Q}_{r'} := \{p \in P_{m,l}^{(f)}(n) : p \equiv r' \pmod{q}\}.$$

Thus we derive from (2.5) that

$$(2.7) \quad \log L_{m,l}^{(f)}(n) = \sum_{r' \in T(q)} \sum_{\substack{p \in P_{m,l}^{(f)}(n) \\ p \equiv r' \pmod{q}}} \log p + O(\sqrt{n}) = \sum_{r' \in T(q)} \sum_{p \in \mathcal{Q}_{r'}} \log p + O(\sqrt{n}).$$

For any given  $r' \in T(q)$ , there is exactly one  $r \in T(q)$  such that  $rr' \equiv 1 \pmod{q}$ . Thus for any given prime  $p \equiv r' \pmod{q}$ , we have  $\langle b_j r \rangle_{a_j} p \equiv \langle b_j r \rangle_{a_j} r' \equiv b_j r r' \equiv b_j \pmod{a_j}$  for each  $1 \leq j \leq 2$ . Since  $\gcd(p, a_j) = 1$  for  $j = 1, 2$ , we can deduce that all the terms divisible by  $p$  in the arithmetic progression  $\{a_j i + b_j\}_{i=1}^{\infty}$  must be of the form  $(a_j k + \langle b_j r \rangle_{a_j})p$ , where  $k \in \mathbb{N}$ . It follows that for each  $1 \leq j \leq 2$  and any prime  $p \in \mathcal{Q}_{r'}$ , we have that  $p | (a_j i + b_j)$  for some  $mn < i \leq ln$  if and only if there is an integer  $i_j \geq 0$

so that  $a_j mn + b_j < (a_j i_j + \langle b_j r \rangle_{a_j})p \leq a_j ln + b_j$ . Therefore, a prime  $p$  congruent to  $r'$  modulo  $q$  is in  $P_{m,l}^{(f)}(n)$  if and only if  $p \nmid (a_1 b_2 - a_2 b_1)$  and either

$$\frac{a_1 mn + b_1}{\langle b_1 r \rangle_{a_1} + a_1 i_1} < p \leq \frac{a_1 ln + b_1}{\langle b_1 r \rangle_{a_1} + a_1 i_1}$$

for some  $i_1 \in \mathbb{N}$ , or

$$\frac{a_2 mn + b_2}{\langle b_2 r \rangle_{a_2} + a_2 i_2} < p \leq \frac{a_2 ln + b_2}{\langle b_2 r \rangle_{a_2} + a_2 i_2}$$

for some  $i_2 \in \mathbb{N}$ . Thus we have by (2.6) that

$$(2.8) \quad \mathcal{Q}_{r'} = \bigcup_{j=1}^2 \bigcup_{i=0}^{\infty} \left\{ \text{prime } p \equiv r' \pmod{q} : \frac{a_j mn + b_j}{\langle b_j r \rangle_{a_j} + a_j i} < p \leq \frac{a_j ln + b_j}{\langle b_j r \rangle_{a_j} + a_j i} \right. \\ \left. \text{and } p \nmid (a_1 b_2 - a_2 b_1) \right\}.$$

To prove Lemma 2.1, we have to treat with the union on the right-hand side of (2.8).

Since  $\gcd(p, a_j) = 1$  for any prime  $p \equiv r' \pmod{q}$ , then by Lemma 3.6 of [9], there is exactly one term divisible by  $p$  in any  $p$  consecutive terms of the arithmetic progression  $\{a_j i + b_j\}_{i=1}^{\infty}$  for each  $1 \leq j \leq 2$ . Therefore, for any prime  $p$  with  $p \leq (l-m)n$  and  $p \equiv r' \pmod{q}$ , there is at least one term divisible by  $p$  in the set  $\{(a_1 i + b_1)(a_2 i + b_2)\}_{i=mn+1}^{ln}$ . Hence we have

$$(2.9) \quad \left\{ \text{prime } p \equiv r' \pmod{q} : p \leq (l-m)n \text{ and } p \nmid (a_1 b_2 - a_2 b_1) \right\} \subseteq \mathcal{Q}_{r'}.$$

By (2.1) and (2.2), for  $j = 1, 2$ , we have that

$$\frac{a_j ln + b_j}{\langle b_j r \rangle_{a_j} + a_j (H_j + 1)} < (l-m)n < \frac{a_j ln + b_j}{\langle b_j r \rangle_{a_j} + a_j H_j}$$

for any positive integer  $n$  with

$$n > n_0 := \left\lfloor \frac{b_j}{(a_j (H_j + 1) + \langle b_j r \rangle_{a_j})(l-m) - a_j l} \right\rfloor.$$

It then follows that for  $j = 1, 2$  and all integers  $i$  with  $i > H_j$ , we have

$$\frac{a_j mn + b_j}{\langle b_j r \rangle_{a_j} + a_j i} < \frac{a_j ln + b_j}{\langle b_j r \rangle_{a_j} + a_j i} < (l-m)n$$

for any positive integer  $n > n_0$ . So we can deduce that

$$\bigcup_{j=1}^2 \bigcup_{i=H_j+1}^{\infty} \left\{ \text{prime } p \equiv r' \pmod{q} : \frac{a_j mn + b_j}{\langle b_j r \rangle_{a_j} + a_j i} < p \leq \frac{a_j ln + b_j}{\langle b_j r \rangle_{a_j} + a_j i} \text{ and } p \nmid (a_1 b_2 - a_2 b_1) \right\} \subseteq \left\{ \text{prime } p \equiv r' \pmod{q} : p \leq (l-m)n \text{ and } p \nmid (a_1 b_2 - a_2 b_1) \right\}$$

for any positive integer  $n > n_0$ . It then follows from (2.8) and (2.9) that

$$(2.10) \quad \mathcal{Q}_{r'} = \left( \bigcup_{j=1}^2 \bigcup_{i=0}^{H_j} \left\{ \text{prime } p \equiv r' \pmod{q} : \frac{a_j mn + b_j}{\langle b_j r \rangle_{a_j} + a_j i} < p \leq \frac{a_j ln + b_j}{\langle b_j r \rangle_{a_j} + a_j i} \right\} \right. \\ \left. \bigcup \left\{ \text{prime } p \equiv r' \pmod{q} : p \leq (l-m)n \right\} \right) \setminus \left\{ \text{prime } p : p \nmid (a_1 b_2 - a_2 b_1) \right\}$$

for any positive integer  $n > n_0$ .

Comparing (2.3) with (2.10) if  $n > n_0$  and comparing (2.3) with (2.8) if  $n \leq n_0$ , we know that there are at most finitely many primes in the union set  $(\mathcal{Q}_{r'} \setminus \mathcal{P}_{r'}) \cup (\mathcal{P}_{r'} \setminus \mathcal{Q}_{r'})$  for any positive integer  $n$ . Therefore

$$(2.11) \quad \sum_{p \in \mathcal{Q}_{r'}} \log p = \sum_{p \in \mathcal{P}_{r'}} \log p + O(\log n).$$

By (2.7) and (2.11), the desired result follows immediately. This concludes the proof of Lemma 2.1.  $\square$

By Lemma 2.1, to estimate  $\log \text{lcm}_{mn < i \leq ln} \{f(i)\}$ , it suffices to estimate  $\sum_{p \in \mathcal{P}_{r'}} \log p$  for each integer  $r'$  satisfying  $1 \leq r' \leq q$  and  $\gcd(r', q) = 1$ , which will be done in the following.

**Lemma 2.2.** *Let  $r'$  and  $r$  be any given integers such that  $1 \leq r', r \leq q$  and  $rr' \equiv 1 \pmod{q}$ . If  $a_1 \langle b_2 r \rangle_{a_2} \geq a_2 \langle b_1 r \rangle_{a_1}$ , then*

$$\sum_{p \in \mathcal{P}_{r'}} \log p = \frac{n}{\varphi(q)} \lambda_r(a_1, a_2, b_1, b_2) + o(n),$$

where  $\mathcal{P}_{r'}$  and  $\lambda_r(a_1, a_2, b_1, b_2)$  are defined as in (2.3) and (1.4), respectively.

*Proof.* Since  $a_1 \langle b_2 r \rangle_{a_2} \geq a_2 \langle b_1 r \rangle_{a_1}$ , we have

$$(2.12) \quad \frac{a_1 l n}{\langle b_1 r \rangle_{a_1} + a_1 i} \geq \frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 i} \text{ and } \frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1 i} \geq \frac{a_2 m n}{\langle b_2 r \rangle_{a_2} + a_2 i}$$

for any integer  $i \geq 0$ . On the other hand, for any integer  $i \geq 0$ , we have

$$(2.13) \quad \frac{a_1 l n}{\langle b_1 r \rangle_{a_1} + a_1(i+1)} < \frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 i} \text{ and } \frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1(i+1)} \leq \frac{a_2 m n}{\langle b_2 r \rangle_{a_2} + a_2 i}$$

since  $0 \leq a_1 \langle b_2 r \rangle_{a_2} - a_2 \langle b_1 r \rangle_{a_1} < a_1 a_2$  and  $l > m \geq 0$ . Let  $K_1 = g_r(a_1, a_2, b_1, b_2)$  and  $K_2 = h_r(a_1, a_2, b_1, b_2)$ . Then by (1.2) and (1.3), we get

$$(2.14) \quad K_1 = \left\lfloor \frac{a_1 a_2 l + a_2 \langle b_1 r \rangle_{a_1} m - a_1 \langle b_2 r \rangle_{a_2} l}{a_1 a_2 (l - m)} \right\rfloor$$

and

$$(2.15) \quad K_2 = \left\lfloor \frac{a_1 \langle b_2 r \rangle_{a_2} m - a_2 \langle b_1 r \rangle_{a_1} l}{a_1 a_2 (l - m)} \right\rfloor.$$

Thus by (1.4), in order to show Lemma 2.2, we only need to prove that

$$(2.16) \quad \sum_{p \in \mathcal{P}_{r'}} \log p = \frac{n}{\varphi(q)} \left( \sum_{i \in S_{K_1}} \frac{a_1 l}{\langle b_1 r \rangle_{a_1} + a_1 i} - \sum_{i \in S_{K_1-1}} \frac{a_2 m}{\langle b_2 r \rangle_{a_2} + a_2 i} + \sum_{i \in S_{K_2}} \left( \frac{a_2 l}{\langle b_2 r \rangle_{a_2} + a_2 i} - \frac{a_1 m}{\langle b_1 r \rangle_{a_1} + a_1 i} \right) \right) + o(n).$$

In the following we show that (2.16) is true. For this purpose, we need to analyze the following union

$$(2.17) \quad \mathcal{T}_r := \left( \bigcup_{j=1}^2 \bigcup_{i=0}^{H_j} \left( \frac{a_j m n}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j l n}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup (0, (l - m)n],$$

since (2.3) gives that

$$(2.18) \quad \mathcal{P}_{r'} = \{\text{prime } p \equiv r' \pmod{q} : p \in \mathcal{T}_r\}.$$

Evidently, we have

$$\frac{a_1 l - (l - m) \langle b_1 r \rangle_{a_1}}{a_1(l - m)} - \frac{a_2 l - (l - m) \langle b_2 r \rangle_{a_2}}{a_2(l - m)} = \frac{a_1 \langle b_2 r \rangle_{a_2} - a_2 \langle b_1 r \rangle_{a_1}}{a_1 a_2}$$

and

$$0 \leq \frac{a_1 \langle b_2 r \rangle_{a_2} - a_2 \langle b_1 r \rangle_{a_1}}{a_1 a_2} < 1.$$

Thus by (2.1) and (2.2) we get that

$$(2.19) \quad H_1 = H_2 \text{ or } H_2 + 1.$$

Moreover, for each  $1 \leq j \leq 2$ , it follows from (2.1) and (2.2) that

$$\frac{a_j m - (l - m) \langle b_j r \rangle_{a_j}}{a_j(l - m)} < H_j \leq \frac{a_j l - (l - m) \langle b_j r \rangle_{a_j}}{a_j(l - m)}.$$

Hence for each  $1 \leq j \leq 2$ ,

$$(2.20) \quad \frac{a_j m n}{\langle b_j r \rangle_{a_j} + a_j H_j} < (l - m)n \leq \frac{a_j l n}{\langle b_j r \rangle_{a_j} + a_j H_j}.$$

By (2.14), we have  $K_1 \geq 0$  and

$$K_1 - 1 \leq \frac{a_1 a_2 m + a_2 \langle b_1 r \rangle_{a_1} m - a_1 \langle b_2 r \rangle_{a_2} l}{a_1 a_2 (l - m)} < K_1.$$

It then follows that

$$(2.21) \quad \frac{a_1 l n}{\langle b_1 r \rangle_{a_1} + a_1(i + 1)} > \frac{a_2 m n}{\langle b_2 r \rangle_{a_2} + a_2 i}$$

for any  $i \geq K_1$  and

$$(2.22) \quad \frac{a_1 l n}{\langle b_1 r \rangle_{a_1} + a_1(i + 1)} \leq \frac{a_2 m n}{\langle b_2 r \rangle_{a_2} + a_2 i}$$

for any  $0 \leq i \leq K_1 - 1$  if  $K_1 \geq 1$ .

From (2.15), we know that  $K_2$  may be smaller than 0, and

$$a_1 a_2 K_2 (l - m) \leq a_1 \langle b_2 r \rangle_{a_2} m - a_2 \langle b_1 r \rangle_{a_1} l < a_1 a_2 (K_2 + 1)(l - m).$$

Thus

$$(2.23) \quad \frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 i} > \frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1 i}.$$

for any  $i \geq \max(0, K_2 + 1)$ , and

$$(2.24) \quad \frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 i} \leq \frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1 i}$$

for any  $0 \leq i \leq K_2$  if  $K_2 \geq 0$ .

For  $j = 1, 2$ , if  $H_j \geq 1$ , then by (2.1) and (2.2) we infer that

$$\begin{aligned} \frac{a_j m n}{\langle b_j r \rangle_{a_j} + a_j(i - 1)} - \frac{a_j l n}{\langle b_j r \rangle_{a_j} + a_j i} &= \frac{a_j (a_j l n - (l - m) \langle b_j r \rangle_{a_j} n - a_j i (l - m) n)}{(\langle b_j r \rangle_{a_j} + a_j(i - 1))(\langle b_j r \rangle_{a_j} + a_j i)} \\ &\geq \frac{a_j (a_j H_j (l - m) n - a_j i (l - m) n)}{(\langle b_j r \rangle_{a_j} + a_j(i - 1))(\langle b_j r \rangle_{a_j} + a_j i)} \\ &= \frac{a_j^2 (H_j - i)(l - m) n}{(\langle b_j r \rangle_{a_j} + a_j(i - 1))(\langle b_j r \rangle_{a_j} + a_j i)} \geq 0 \end{aligned}$$

for any integer  $i$  with  $1 \leq i \leq H_j$ , which means that

$$\frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j(i-1)} \geq \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i}$$

for any integer  $i$  with  $1 \leq i \leq H_j$ . Hence for  $j = 1, 2$ , the intersection

$$(2.25) \quad \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i_1}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i_1} \right] \cap \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i_2}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i_2} \right]$$

is empty for any  $0 \leq i_1 \neq i_2 \leq H_j$  if  $H_j \geq 1$ . Now we consider the following two cases.

CASE 1.  $K_1 \geq K_2 + 1$ . First, it is easy to see from (2.2), (2.14) and (2.15) that  $K_1 \leq H_2$  and  $K_2 + 1 \leq H_2$ . For any integer  $i \geq \max(0, K_2 + 1)$ , we have by (2.12) and (2.23) that

$$(2.26) \quad \bigcup_{j=1}^2 \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] = \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 i} \right].$$

It then follows from (2.19)-(2.21) and (2.26) that

$$(2.27) \quad \begin{aligned} & \left( \bigcup_{j=1}^2 \bigcup_{i=K_1}^{H_j} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup (0, (l-m)n] \\ &= \left( \bigcup_{i=K_1}^{H_2} \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 i} \right] \right) \cup (0, (l-m)n] \\ & \quad \cup \left( \frac{a_1 mn}{\langle b_1 r \rangle_{a_1} + a_1 H_1}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 H_1} \right] \\ &= \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 H_2}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \right] \cup \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 H_1} \right] \\ &= \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \right]. \end{aligned}$$

Thus we can derive from (2.17) and (2.27) that

$$\begin{aligned} \mathcal{T}_r &= \bigcup_{j=1}^2 \left( \bigcup_{i \in S_{K_1-1}} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \cup \right. \\ & \quad \left. \bigcup_{i=K_1}^{H_j} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup (0, (l-m)n] \\ &= \left( \bigcup_{j=1}^2 \bigcup_{i \in S_{K_2}} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \right] \\ & \quad \cup \left( \bigcup_{i \in S_{K_1-1} \setminus S_{K_2}} \bigcup_{j=1}^2 \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right). \end{aligned}$$



It then follows from (2.26) that

$$(2.28) \quad \mathcal{T}_r = \left( \bigcup_{j=1}^2 \bigcup_{i \in S_{K_2}} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup \\ \left( \bigcup_{i \in S_{K_1-1} \setminus S_{K_2}} \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 i} \right] \right) \cup \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \right].$$

Note that  $S_{K_2}$  is empty if  $K_2 < 0$ , and  $S_{K_1-1} \setminus S_{K_2}$  is empty if  $K_1 = K_2 + 1$  or  $K_1 = 0$ . By (2.22), we know that the following union

$$\bigcup_{i \in S_{K_1-1} \setminus S_{K_2}} \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 i} \right]$$

is a disjoint union. But by (2.22), (2.24) and (2.25), the union

$$\bigcup_{j=1}^2 \bigcup_{i \in S_{K_2}} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right]$$

is a disjoint union. Therefore by (2.22), the union on the right-hand side of (2.28) is disjoint. Thus applying (2.18), (2.28) and prime number theorem for arithmetic progressions (see, for example [14]), we obtain that

$$\begin{aligned} \sum_{p \in \mathcal{P}_{r'}} \log p &= \sum_{j=1}^2 \sum_{i \in S_{K_2}} \sum_{\substack{\frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i} < p \leq \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \\ p \equiv r' \pmod{q}}} \log p + \sum_{\substack{p \leq \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \\ p \equiv r' \pmod{q}}} \log p \\ &+ \sum_{i \in S_{K_1-1} \setminus S_{K_2}} \sum_{\substack{\frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i} < p \leq \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 i} \\ p \equiv r' \pmod{q}}} \log p \\ &= \frac{n}{\varphi(q)} \left( \sum_{j=1}^2 \sum_{i \in S_{K_2}} \left( \frac{a_j l}{\langle b_j r \rangle_{a_j} + a_j i} - \frac{a_j m}{\langle b_j r \rangle_{a_j} + a_j i} \right) + \frac{a_1 l}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \right. \\ &\quad \left. + \sum_{i \in S_{K_1-1} \setminus S_{K_2}} \left( \frac{a_1 l}{\langle b_1 r \rangle_{a_1} + a_1 i} - \frac{a_2 m}{\langle b_2 r \rangle_{a_2} + a_2 i} \right) \right) + o(n). \end{aligned}$$

Then (2.16) follows immediately. So (2.16) is proved for Case 1.

CASE 2.  $K_1 \leq K_2$ . Then by (2.14) and (2.15), we have  $K_2 \geq K_1 \geq 0$ . If  $K_2 + 1 \leq H_2$ , applying (2.19)-(2.21) and (2.26), one infers that

$$\begin{aligned}
(2.29) \quad & \bigcup_{j=1}^2 \bigcup_{i=K_2+1}^{H_j} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \cup (0, (l-m)n] \\
&= \bigcup_{i=K_2+1}^{H_2} \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 i} \right] \cup (0, (l-m)n] \\
&\quad \cup \left( \frac{a_1 mn}{\langle b_1 r \rangle_{a_1} + a_1 H_1}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 H_1} \right] \\
&= \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 H_2}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 (K_2 + 1)} \right] \cup \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 H_1} \right] \\
&= \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 (K_2 + 1)} \right].
\end{aligned}$$

Hence by (2.17) and (2.29),

$$(2.30) \quad \mathcal{T}_r = \left( \bigcup_{j=1}^2 \bigcup_{i=0}^{K_2} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 (K_2 + 1)} \right].$$

Moreover, we have

$$\begin{aligned}
(2.31) \quad & \bigcup_{j=1}^2 \bigcup_{i=0}^{K_2} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \\
&= \left( \bigcup_{j=1}^2 \bigcup_{i \in S_{K_1-1}} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup \left( \frac{a_1 mn}{\langle b_1 r \rangle_{a_1} + a_1 K_1}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \right] \cup \\
&\quad \bigcup_{i \in S_{K_2-1} \setminus S_{K_1-1}} \left( \left( \frac{a_1 mn}{\langle b_1 r \rangle_{a_1} + a_1 (i+1)}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 (i+1)} \right] \cup \right. \\
&\quad \left. \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i}, \frac{a_2 ln}{\langle b_2 r \rangle_{a_2} + a_2 i} \right] \right) \cup \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 K_2}, \frac{a_2 ln}{\langle b_2 r \rangle_{a_2} + a_2 K_2} \right].
\end{aligned}$$

But since  $K_2 \geq 0$  and  $K_2 \geq K_1$ , applying (2.13) and (2.21) gives us that

$$(2.32) \quad \left( 0, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 (K_2 + 1)} \right] \cup \left( \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 K_2}, \frac{a_2 ln}{\langle b_2 r \rangle_{a_2} + a_2 K_2} \right] = \left( 0, \frac{a_2 ln}{\langle b_2 r \rangle_{a_2} + a_2 K_2} \right].$$

Therefore by (2.13), (2.21) and (2.30)-(2.32), we have

$$\begin{aligned}
(2.33) \quad & \mathcal{T}_r = \left( \bigcup_{j=1}^2 \bigcup_{i \in S_{K_1-1}} \left( \frac{a_j mn}{\langle b_j r \rangle_{a_j} + a_j i}, \frac{a_j ln}{\langle b_j r \rangle_{a_j} + a_j i} \right] \right) \cup \left( \frac{a_1 mn}{\langle b_1 r \rangle_{a_1} + a_1 K_1}, \frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \right] \\
&\quad \left( \bigcup_{i \in S_{K_2-1} \setminus S_{K_1-1}} \left( \frac{a_1 mn}{\langle b_1 r \rangle_{a_1} + a_1 (i+1)}, \frac{a_2 ln}{\langle b_2 r \rangle_{a_2} + a_2 i} \right] \right) \cup \left( 0, \frac{a_2 ln}{\langle b_2 r \rangle_{a_2} + a_2 K_2} \right].
\end{aligned}$$

Note that  $S_{K_2-1} \setminus S_{K_1-1}$  is empty if  $K_1 = K_2$ , and  $S_{K_1-1}$  is empty if  $K_1 = 0$ . By (2.22), we have that

$$\frac{a_1 l n}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \leq \frac{a_2 m n}{\langle b_2 r \rangle_{a_2} + a_2 (K_1 - 1)}$$

if  $K_1 \geq 1$ . Using (2.24), one has

$$\frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 K_2} \leq \frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1 K_2} \text{ and } \frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 K_1} \leq \frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1 K_1}.$$

Then using (2.22), (2.24) and (2.25), we derive that any two intervals in the union on the right-hand side of (2.33) are disjoint. Hence we have by (2.18) and (2.33) that

$$\begin{aligned} \sum_{p \in \mathcal{P}_{r'}} \log p &= \sum_{j=1}^2 \sum_{i \in S_{K_1-1}} \sum_{\substack{\frac{a_j m n}{\langle b_j r \rangle_{a_j} + a_j i} < p \leq \frac{a_j l n}{\langle b_j r \rangle_{a_j} + a_j i} \\ p \equiv r' \pmod{q}}} \log p + \sum_{\substack{p \leq \frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 K_2} \\ p \equiv r' \pmod{q}}} \log p \\ &+ \sum_{\substack{\frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1 K_1} < p \leq \frac{a_1 l n}{\langle b_1 r \rangle_{a_1} + a_1 K_1} \\ p \equiv r' \pmod{q}}} \log p + \sum_{i \in S_{K_2-1} \setminus S_{K_1-1}} \sum_{\substack{\frac{a_1 m n}{\langle b_1 r \rangle_{a_1} + a_1 (i+1)} < p \leq \frac{a_2 l n}{\langle b_2 r \rangle_{a_2} + a_2 i} \\ p \equiv r' \pmod{q}}} \log p. \end{aligned}$$

It then follows from the prime number theorem for arithmetic progressions that

$$\begin{aligned} \sum_{p \in \mathcal{P}_{r'}} \log p &= \frac{n}{\varphi(q)} \left( \sum_{j=1}^2 \sum_{i \in S_{K_1-1}} \left( \frac{a_j l}{\langle b_j r \rangle_{a_j} + a_j i} - \frac{a_j m}{\langle b_j r \rangle_{a_j} + a_j i} \right) + \frac{a_2 l}{\langle b_2 r \rangle_{a_2} + a_2 K_2} \right. \\ &\quad \left. + \frac{a_1 (l - m)}{\langle b_1 r \rangle_{a_1} + a_1 K_1} + \sum_{i \in S_{K_2-1} \setminus S_{K_1-1}} \left( \frac{a_2 l}{\langle b_2 r \rangle_{a_2} + a_2 i} - \frac{a_1 m}{\langle b_1 r \rangle_{a_1} + a_1 (i+1)} \right) \right) + o(n) \\ &= \frac{n}{\varphi(q)} \left( \sum_{i \in S_{K_1}} \frac{a_1 l}{\langle b_1 r \rangle_{a_1} + a_1 i} - \sum_{i \in S_{K_1-1}} \frac{a_2 m}{\langle b_2 r \rangle_{a_2} + a_2 i} \right. \\ &\quad \left. + \sum_{i \in S_{K_2}} \left( \frac{a_2 l}{\langle b_2 r \rangle_{a_2} + a_2 i} - \frac{a_1 m}{\langle b_1 r \rangle_{a_1} + a_1 i} \right) \right) + o(n) \end{aligned}$$

as required. Thus (2.16) is true for Case 2.

This completes the proof of Lemma 2.2.  $\square$

### 3. Proof of Theorem 1.1

In this section, we use the results presented in Section 2 to give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let  $r'$  and  $r$  be any given integers such that  $1 \leq r', r \leq q$  and  $rr' \equiv 1 \pmod{q}$ . Exchanging  $a_1$  with  $a_2$  and  $b_1$  with  $b_2$  simultaneously,  $f(x) = (a_2 x + b_2)(a_1 x + b_1)$  is unchanged, meanwhile the condition  $a_1 \langle b_2 r \rangle_{a_2} \geq a_2 \langle b_1 r \rangle_{a_1}$  in Lemma 2.2 becomes  $a_2 \langle b_1 r \rangle_{a_1} \geq a_1 \langle b_2 r \rangle_{a_2}$ , and in the conclusion of Lemma 2.2,  $\lambda_r(a_1, a_2, b_1, b_2)$  becomes  $\lambda_r(a_2, a_1, b_2, b_1)$ . Thus by Lemma 2.2, we obtain that

$$\sum_{p \in \mathcal{P}_{r'}} \log p = \frac{n}{\varphi(q)} \lambda_r(a_2, a_1, b_2, b_1) + o(n)$$

if  $a_2 \langle b_1 r \rangle_{a_1} \geq a_1 \langle b_2 r \rangle_{a_2}$ . Note that if  $a_1 \langle b_2 r \rangle_{a_2} = a_2 \langle b_1 r \rangle_{a_1}$ , then

$$\frac{a_1 mn}{\langle b_1 r \rangle_{a_1} + a_1 i} = \frac{a_2 mn}{\langle b_2 r \rangle_{a_2} + a_2 i}$$

and

$$\frac{a_1 ln}{\langle b_1 r \rangle_{a_1} + a_1 i} = \frac{a_2 ln}{\langle b_2 r \rangle_{a_2} + a_2 i}$$

for any integer  $i \geq 0$ . Moreover, one has by (1.2) and (1.3) that

$$g_r(a_1, a_2, b_1, b_2) = g_r(a_2, a_1, b_2, b_1)$$

and

$$h_r(a_1, a_2, b_1, b_2) = h_r(a_2, a_1, b_2, b_1)$$

if  $a_1 \langle b_2 r \rangle_{a_2} = a_2 \langle b_1 r \rangle_{a_1}$ . It then follows from (1.4) that

$$\lambda_r(a_1, a_2, b_1, b_2) = \lambda_r(a_2, a_1, b_2, b_1)$$

if  $a_1 \langle b_2 r \rangle_{a_2} = a_2 \langle b_1 r \rangle_{a_1}$ .

Now by Lemma 2.2 and the above discussion, we get that

$$(3.1) \quad \sum_{p \in \mathcal{P}_{r'}} \log p = \frac{n}{\varphi(q)} A_r + o(n),$$

where  $A_r$  is defined as in (1.5). Since  $rr' \equiv 1 \pmod{q}$  and  $1 \leq r', r \leq q$ ,  $r$  runs over the set of all positive integers no more than  $q$  that are relatively prime to  $q$  as  $r'$  does, it then follows from (3.1) and Lemma 2.1 that

$$\begin{aligned} & \log \text{lcm}_{mn < i \leq ln} \{f(i)\} \\ &= \frac{n}{\varphi(q)} \sum_{\substack{r'=1 \\ \gcd(r', q)=1}}^q A_r + o(n) \\ &= \frac{n}{\varphi(q)} \sum_{\substack{r=1 \\ \gcd(r, q)=1}}^q A_r + o(n) \end{aligned}$$

as desired. This finishes the proof of Theorem 1.1.  $\square$

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